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MONDAY, JANUARY 22ND, 1855.

LIEUT.-COL. LARCOM, F.R.S., VICE-PRESIDENT,
in the Chair.

REV. PROFESSOR GRAVES, D.D., read a Paper on the solution of the equation of Laplace's functions.

"Mr. Carmichael was the first to observe that the partial differential equation of the second order,

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=0,$$

or

$$D_1^2V+D_2^2V+D_3^2V=0, \tag{1}$$

known as the equation of Laplace's functions, may be reduced, by means of Sir William Hamilton's imaginaries, to the symbolic form,

$$(D_1+jD_2+kD_3)\,(D_1-jD_2-kD_3)\,V=0.$$

Its complete solution is, therefore, the sum of those of the two equations of the first order,

$$(D_1+jD_2+kD_3)\,V=0,$$
$$(D_1-jD_2-kD_3)\,V=0;$$

and these latter solutions have been presented to us by Mr. Carmichael in the symbolic forms,

$$V=e^{-x(jD_2+kD_3)}f_1\,(y,z), \tag{2}$$
$$V=e^{x(jD_2+kD_3)}f_2\,(y,z), \tag{3}$$

in which f_1 and f_2 stand for quite arbitrary functions. Following, however, too closely the analogies of ordinary algebra, Mr. Carmichael has fallen into an error in interpreting the right-hand members of these formulæ. He has made

$$e^{-x(jD_2+kD_3)}f_1(y, z) = f_1(y-jx,\ z-kx), \qquad (4)$$

$$e^{x(jD_2+kD_3)}f_2(y, z) = f_2(y+jx,\ z+kx). \qquad (5)$$

Sir William Hamilton at once perceived the inaccuracy of these results, and referred it to its origin, which was the erroneous supposition, that

$$e^{jD_2+kD_3} = e^{jD_2}\,e^{kD_3}.$$

This last equivalence does *not* subsist, because the symbols j and k are not commutative.

" Indeed, the consideration of a simple case might lead to the suspicion that the formula (4) was incorrect. Suppose that $f_1(y, z) = yz$: it becomes at once a question, what is the meaning of $f_1(y-jx,\ z-kx)$? Is it $(y-jx)(z-kx)$, or $(z-kx)(y-jx)$? for these expressions have different values. Thus, in the first instance, it is apparent that the assigned result is ambiguous. But from what follows it will appear that neither $(y-jx)(z-kx)$, nor $(z-kx)(y-jx)$, is equivalent to

$$e^{-x(jD_2+kD_3)}yz.$$

" The question relative to the interpretation of the symbolic formulæ (2) and (3) being in this state, I have endeavoured to resolve it in the present Paper.

" As a first step in our investigation, let us inquire what is the effect of the symbol,

$$e^{x(jD_2+kD_3)},$$

or π, as it will be more convenient to denote it, upon the term $y^m z^n$, m and n being positive integers.

" Beginning with simple cases, we shall find by actual expansion of the exponential symbol,

1. $\pi\ y^m = (y+jx)^m$, and $\pi\ z^n = (z+kx)^n$.
2. $\pi\ yz = yz + jxz + kxy$.
3. $\pi\ y^2z = y^2z + 2jxyz + kxy^2 - x^2z - \frac{1}{3}kx^3$.
4. $\pi\ y^2z^3 = y^2z^3 + 2jxyz^3 + 3kxy^2z^2 - x^2z^3 - 3x^2y^2z - kx^3z^2$
$- 2jx^3yz - kx^3y^2 + x^4z + \frac{1}{5}kx^5$.

The first of these results shows that the effect of π upon any function of y alone is to change it into the like function of $y+jx$; and the effect of the same symbol upon any function of z alone is to change it into the like function of $z+kx$.

“But the second example shows that its effect upon yz is to change it, not into either $(y+jx)\ (z+kx)$, or $(z+kx)\ (y+jx)$, but into half the sum of these different expressions. For

$$\tfrac{1}{2}\{(y+jx)\ (z+kz)+(z+kx)\ (y+jx)\}=yz+jxz+kxy,$$

in virtue of the equation $jk+kj=0$.

“Again, the third example shows that the effect of π upon y^2z is to change it, not into any one of the three different expressions,

$$(y+jx)^2\ (z+kz),\ \ (y+jx)\ (z+kx)\ (y+jx),\ \text{ and }\ (z+kx)\ (y+jx)^2,$$

but into the third part of the sum of the three. It is easy to see that this result follows from the equations,

$$j^2=-1,\quad k^2=-1,\quad jk+kj=0.$$

“Pursuing the same course we shall find that the effect of π upon y^2z^3 is to change it into the tenth part of the sum of the ten expressions,

$$\begin{array}{l}
(y+jx)^2\ (z+kx)^3,\\
(y+jx)\ (z+kx)\ (y+jx)\ (z+kx)^2,\\
(y+jx)\ (z+kx)^2\ (y+jx)\ (z+kx),\\
(y+jx)\ (z+kx)^3\ (y+jx),\\
(z+kx)\ (y+jx)^2\ (z+kx)^2,\\
(z+kx)\ (y+jx)\ (z+kx)\ (y+jx)\ (z+kx),\\
(z+kx)\ (y+jx)\ (z+kx)^2\ (y+jx),\\
(z+kx)^2\ (y+jx)^2\ (z+kx),\\
(z+kx)^2\ (y+jx)\ (z+kx)\ (y+jx),\\
(z+kx)^3\ (y+jx)^2,
\end{array}$$

which arise as the differently arranged products of the five factors, of which two are equal to $y+jx$, and three to $z+kx$.

“Following up the analogy, we are led to expect that the effect of π upon $y^m z^n$ will be to change it into the

$\left\{\dfrac{(m+n)\,!}{m!\,n!}\right\}^{th}$ part of the sum of all the $\dfrac{(m+n)\,!}{m!\,n!}$ different expressions which arise as the differently arranged products of the $m+n$ factors of which m are equal to $y+jx$, and n to $z+kx$.

"The following reasoning demonstrates the truth of the proposition just stated.

"Let C be the coefficient of $x^{\mu+\nu}y^{m-\mu}z^{n-\nu}$ in the development of $\pi\, y^m z^n$. Then C will be equal to the coefficient of $x^{\mu+\nu}D^2_\mu D^3_\nu$ in the development of

$$e^{x(jD_2+kD_3)},$$

multiplied by

$$m\,(m-1)\,.\,.\,.\,.\,.\,(m-\mu+1)\,n\,(n-1)\,.\,.\,.\,.\,.\,(n-\nu+1).$$

But, in the development of the exponential, $D_2^\mu D_3^\nu$ occurs only in the term

$$\frac{x^{\mu+\nu}(jD_2+kD_3)^{\mu+\nu}}{(\mu+\nu)\,!}:$$

and there has for its coefficient

$$\frac{\Sigma\,(\mu,\nu)}{(\mu+\nu)\,!};$$

the numerator $\Sigma\,(\mu,\nu)$ denoting the sum of all the variously arranged products, into each of which enter $\mu\, j$ s, and $\nu\, k$ s. Consequently, we have

$$C=\frac{m\,!\,n\,!}{(m-\mu)\,!\,(n-\nu)\,!\,(\mu+\nu)\,!}\,\Sigma\,(\mu,\nu).$$

"But again, C_1 the coefficient of $x^{\mu+\nu}y^{m-\mu}z^{n-\nu}$ in the $\left\{\dfrac{(m+n)!}{m!\,n!}\right\}^{th}$ part of the sum of the $\dfrac{(m+n)!}{m!\,n!}$ differently arranged products of the $m+n$ factors, of which m are equal to $y+jx$, and n to $z+kx$, will be equal to

$$\frac{m!\,n!}{(m+n)!}\,S,$$

where S denotes the sum of all the differently arranged products of $m+n$ factors, of which $m-\mu$ are y s, $n-\nu$ are z s, μ are j s, and ν are k s. Now the *number* of these arrangements in S is

$$\frac{(m+n)!}{(m-\mu)!\,(n-\nu)!\,\mu!\,\nu!},$$

and S itself will obviously be of the form $N\Sigma(\mu, \nu)$, N being some numerical coefficient depending upon m, n, μ, and ν. But as the *number* of differently arranged terms in $\Sigma(\mu, \nu)$ is

$$\frac{(\mu+\nu)!}{\mu!\,\nu!},$$

it is plain that we shall have

$$N=\frac{(m+n)!}{(m-\mu)!\,(n-\nu)!\,(\mu+\nu)!},$$

and consequently,

$$C_1=\frac{m!\,n!}{(m-\mu)!\,(n-\nu)!\,(\mu+\nu)!}\,\Sigma(\mu, \nu).$$

"Thus, we have found that $C=C_1$, and as this is true for the numerical coefficient of every term in the development of $\pi\, y^m z^n$, we are warranted in concluding that this latter expression is equal to the

$$\left\{\frac{(m+n)!}{m!\,n!}\right\}^{th} \text{part of the sum of all the } \frac{(m+n)!}{m!\,n!}$$

differently arranged products of the $m+n$ factors, of which m are equal to $y+jx$, and n to $z+kx$.

"The statement of this theorem, and of other similar ones, may be rendered simpler and more elegant by our assigning a name and symbol to the last-mentioned expression. I propose to call it the *mean value of the product* of the factors combined in different orders: and for the present to denote it by the symbol

$$M(y\overset{m}{+}jx,\ z\overset{n}{+}kx).$$

We may now proceed to interpret the expression

$$e^{x(jD_2+kD_3)}f(y, z),$$

in which $f(y, z)$ is supposed to be of the form

$$\Sigma\, A\, y^m z^n,$$

m and n, as before, being positive integers.

"The exponental symbol being distributive in its nature, we shall have the proposed expression equal to the sum of the mean values of the products corresponding to the several terms such as $Ay^m z^n$. Consequently,

$$e^{x(jD_2+kD_3)}f(y, z) = \Sigma\, AM\,(\overset{m}{y+jx},\ \overset{n}{z+kx}),$$

and, with an interpretation suggested by what has been already said, we may write finally,

$$e^{x(jD_2+kD_3)}f(y, z) = Mf(y+jx,\ z+kx).$$

"The boundaries of algebra having been of late extended so as to include symbols which are not commutative with each other, it becomes absolutely necessary to have the means of denoting certain standard and constantly occurring combinations in brief and unambiguous ways. The symbol M, proposed in this paper, may perhaps be a useful contribution to mathematical language. It has the recommendation of having been already used in a similar, though less extensive, meaning by M. Cauchy. It may also be regarded as an extension of Sir William Hamilton's notation of $S\,(\alpha, \beta)$, which stands in the Calculus of Quaternions for $\frac{1}{2}\,(\alpha\beta + \beta\alpha)$.

"Knowing how to interpret the expression,

$$e^{x(jD_2+kD_3)},$$

we are enabled, in general, to solve the equation,

$$D_1^2 V + D_2^2 V + D_3^2 V = U, \tag{6}$$

in which U denotes any function expressed by means of positive and integer powers of x, y, and z. The solution depends upon our being able to invert the operations denoted by

$$D_1 + jD_2 + kD_3, \text{ and } D_1 - jD_2 - kD_3\,;$$

and as these inverse operations are respectively,

$$e^{-x(jD_2+kD_3)}D_1^{-1}e^{x(jD_2+kD_3)}, \text{ or } \pi^{-1}D_1^{-1}\pi,$$

and

$$e^{x(jD_2+kD_3)}D_1^{-1}e^{-x(jD_2+kD_3)}, \text{ or } \pi\ D_1^{-1}\pi^{-1},$$

we shall have

$$V = \pi D_1^{-1}\pi^{-2}D_1^{-1}\pi U + Mf_1(y-jx_1,\ z-kx) + Mf_2(y+jx,\ z+kx),$$

the two latter terms being the solution of (1). This complete solution, when developed, appears, in general, in the form,

$$F_1 + jF_2 + kF_2,$$

F_1, F_2, and F_3, being different functions of x, y, z, which singly satisfy the proposed equation.

"For instance, we have seen above that

$$\begin{aligned} e^{x(jD_2+kD_3)}y^2z^3 = y^2z^3 - x^2(z^3+3y^2z) + x^4z, \\ +2j(xyz^3+x^3yz), \\ +k(3xy^2z^2 - x^3z^2 - x^3y^2 + \tfrac{1}{5}x^5). \end{aligned}$$

It will be found on trial, that each line in the right-hand member will by itself satisfy the equation of Laplace's functions.

"The conclusions already obtained may be further generalized. For the equation,

$$\frac{d^2V}{dw^2} + \frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = U,$$

in which U is a function of w, x, y, and z, may be reduced to the symbolic form,

$$(D + iD_1 + jD_2 + kD_3)(D - iD_1 - jD_2 - kD_3)V = U,$$

the solution of which depends on the inversion of the operators,

$$D + iD_1 + jD_2 + kD_3, \text{ and } D - iD_1 - jD_2 - kD_3.$$

Putting

$$iD_1 + jD_2 + kD_3 = \triangleright,$$

a notation employed by Sir William Hamilton, we shall have

$$(D+\triangleright)^{-1} = e^{-w\triangleright} D^{-1} e^{w\triangleright},$$
$$(D-\triangleright)^{-1} = e^{w\triangleright} D^{-1} e^{-w\triangleright}.$$

Consequently,

$$V = e^{w\triangleright} D^{-1} e^{-2w\triangleright} D^{-1} e^{w\triangleright} U$$
$$+ e^{-w\triangleright} f_1(x, y, z) + e^{w\triangleright} f_2(x, y, z), \qquad (7)$$

and to interpret this we must ascertain what is the effect of the symbol

$$e^{w\triangleright},$$

upon any function of x, y, and z.

"Reasoning as before, we should find that its effect upon a term $x^l y^m z^n$, l, m, and n being positive integers, will be to change it into the

$$\left\{\frac{(l+m+n)!}{l!\,m!\,n!}\right\}^{th} \text{part of the sum of all the } \frac{(l+m+n)!}{l!\,m!\,n!}$$

differently arranged products of the $l+m+n$ factors, of which l are equal to $x+iw$, m to $y+jw$, and n to $z+kw$. In other words, to change it into the *mean value* of this product, that is, into

$$M(x \overset{l}{+} iw,\ y \overset{m}{+} jw,\ z \overset{n}{+} kw):$$

and, more generally, the effect of

$$e^{w\triangleright}$$

upon $f(x, y, z)$, any function consisting of positive and integer powers of x, y, and z, will be to change it into

$$Mf(x+iw,\ y+jw,\ z+kw).$$

We are thus enabled to interpret the formula (7), when U, f_1, and f_2, are functions of positive and integer powers. To that case the demonstrations given in this paper are essentially limited. I hope to be able to lay before the Academy the investigation of the cases where l, m, and n, are negative or fractional. So far as I have yet discussed them, they seem to present results which it is more difficult to express with elegance.

"In conclusion, I may be permitted to state some theorems at which I arrived whilst discussing the subject of the present paper. In fact, I at first imagined that the proof of them was necessary to my purpose. They are obtained as follows:—

"In virtue of the laws of combination of the imaginaries i, j, k, we have

$$(ia + jb + kc)^{2\lambda+2\mu+2\nu} = (-1)^{\lambda+\mu+\nu} (a^2 + b^2 + c^2)^{\lambda+\mu+\nu}. \qquad (8)$$

Now, the coefficient of $a^{2\lambda} b^{2\mu} c^{2\nu}$ in the left-hand member of this equation is $\Sigma(2\lambda, 2\mu, 2\nu)$, in conformity with the notation explained in p. 165: and the same coefficient in the right-hand member is plainly

$$(-1)^{\lambda+\mu+\nu} \frac{(\lambda + \mu + \nu)!}{\lambda!\, \mu!\, \nu!}.$$

Consequently, we have the theorem I.

$$\Sigma(2\lambda, 2\mu, 2\nu) = (-1)^{\lambda+\mu+\nu} \frac{(\lambda + \mu + \nu)!}{\lambda!\, \mu!\, \nu!}.$$

"Multiplying both sides of the equation (8) by $ia + jb + kc$, we get

$$(ia + jb + kc)^{2\lambda+1+2\mu+2\nu} = (-1)^{\lambda+\mu+\nu} (ia + jb + kc)(a^2 + b^2 + c^2)^{\lambda+\mu+\nu}.$$

The coefficient of $a^{2\lambda+1} b^{2\mu} c^{2\nu}$ in the left-hand member is $\Sigma(2\lambda + 1, 2\mu, 2\nu)$; and the same coefficient in the right-hand member is

$$(-1)^{\lambda+\mu+\nu} \frac{(\lambda + \mu + \nu)!}{\lambda!\, \mu!\, \nu!}\, i.$$

We have, therefore, II.,

$$\Sigma(2\lambda + 1, 2\mu, 2\nu) = (-1)^{\lambda+\mu+\nu} \frac{(\lambda + \mu + \nu)!}{\lambda!\, \mu!\, \nu!}\, i,$$

and similar expressions for j and k.

"Again, multiplying (8) by $(ia + jb + kc)^2 = -(a^2 + b^2 + c^2)$, we get

$$(ia + jb + kc)^{2\lambda+1+2\mu+1+2\nu} = (-1)^{\lambda+\mu+\nu+1} (a^2 + b^2 + c^2)^{\lambda+\mu+\nu+1}.$$

The coefficient of $a^{2\lambda+1} b^{2\mu+1} c^{2\nu}$ in the left-hand member is $\Sigma(2\lambda + 1, 2\mu + 1, 2\nu)$; whilst in the development of the right-hand member no such term appears, as all the exponents of a, b, c, must be even numbers. We have, therefore, III.,

$$\Sigma\,(2\lambda + 1,\ 2\mu + 1,\ 2\nu) = 0,$$

If $\lambda = 0$, we must put 1 in place of $\lambda\,!$ in the preceding formulæ."

Sir William R. Hamilton made some remarks on Professor Graves' Paper.

Dr. Aquilla Smith presented the following donation from the Archdeacon of Clonmacnoise:

1. Coloured window glass, seventeen fragments:—one fragment in lead fitting; two fragments of lead fittings for glass: from the Abbey of St. Peter and St. Paul, Newtown, Trim.
2. Oval silver reliquary and medal.
3. Ten Italian medals in brass and copper.
4. Two bullæ; a figure representing St. Patrick; a weight stamped with a heart and the letters J. B.; a square ingot or weight; and a small cup: all made of lead.
5. Two silver, one gold, and one brass brooch; one bead made of white glass, with blue stripes.
6. Three silver and four brass finger rings; four rings, iron and brass.
7. One silver and nine brass buckles, of different patterns.
8. Three very small spoons, and two fragments of large spoons.
9. A coat button with glass centre.
10. Two ornaments and harp pin, made of brass.
11. Two pins and portions of a comb, made of bone.
12. One brass and seven iron keys.
13. One old matchlock and fragment of iron chain.
14. One small padlock and stopper for key-hole of safe.
15. One pommel of sword and arrow-head; two knives and one large fork; made of iron.